What Makes Frogs Fly?

A Look into Electromagnetic Levitation, How It Works and How We Use It

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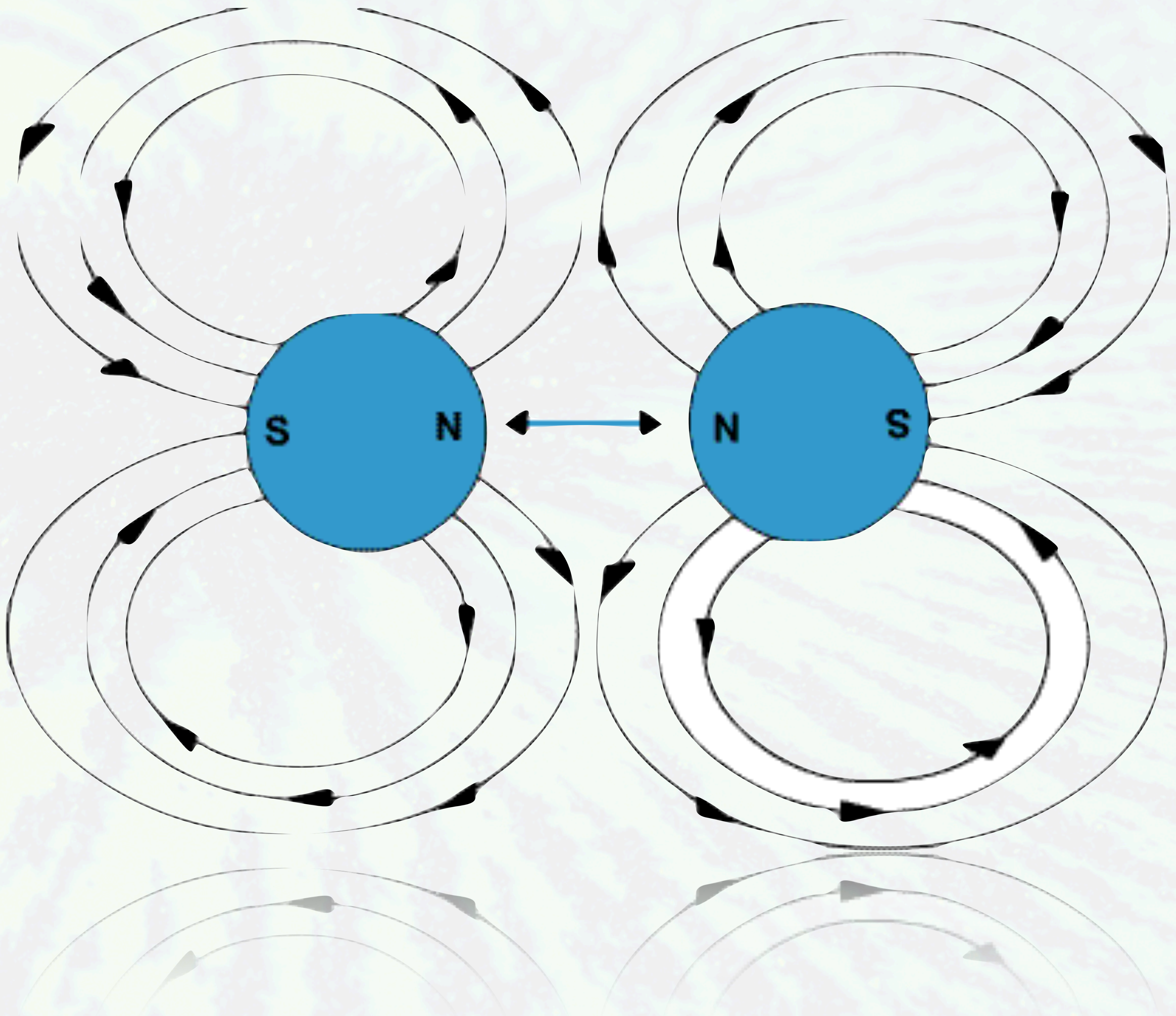
Ever try to balance one magnet atop another?

(Hint: it doesn't work)



Why?

- In theory, if we balance the forces, is it possible?
- Is it similar to balancing a pencil on its tip?
- Can we prove that is <u>*isn't*</u> possible?

What do we mean by levitation?

“Levitation is a state in which a body with a mass m is suspended in stable equilibrium without contact of another body with mass M, with both attractive and repulsive forces acting on both bodies.”

What do we mean by levitation?

"Electromagnetic levitation is a state in which a body with mass m *and either a charge* q *or magnetisation* $\overrightarrow{M}$ *or both is suspended in stable equilibrium without contact of another body with mass* M*, such that the repulsive forces are either magnetostatic or electrostatic or both, and some attractive force exists between the two bodies."*

To the Whiteboard!

Maxwell's Equations

We recall that Maxwell's equations are the following:

Gauss' Law

$$\nabla \cdot \overrightarrow{E} = \frac{\rho}{\varepsilon_0}$$

Gauss' Law for magnetism

$$\nabla \cdot \overrightarrow{B} = 0$$

Maxwell-Faraday Equation

$$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

Ampère's Circuital Law

$$\nabla \times \overrightarrow{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} \right)$$

Stability Conditions

- For equilibrium, we require that the net force on the levitating object to be zero at the equilibrium point. In this case, for a purely electrostatic system this means:

$$\overrightarrow{E} = 0$$

- We can relate this to another value, the potential energy:

$$\overrightarrow{E} = -\nabla U$$

- As a reminder: $\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$

Stability Conditions

- This condition is necessary, but for ***stable*** equilibrium, we require the equilibrium to be at a stable point, thus giving our next condition:

$$\nabla^2 U > 0$$

- Can you remember the name of this operation?
- What does this imply?

Stability Conditions

- A point can follow this condition and not be fully stable.
- If we compare the inside of a bowl to a Pringle, the centre of both follows $\nabla^2 U > 0$, however one is semi-stable and one is stable.

- In short, we require the point to be concave up in all directions. Thus, this condition is also necessary, but not completely sufficient for stable equilibrium.
  - We have a tool for this: *The Hessian Matrix*

The Hessian Matrix

- The Hessian matrix is given as:

$$\mathbf{H}(U) = \begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial z \partial x} \\ \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial z \partial y} \\ \frac{\partial^2 U}{\partial x \partial z} & \frac{\partial^2 U}{\partial y \partial z} & \frac{\partial^2 U}{\partial z^2} \end{bmatrix}$$

- We can see that the trace of this matrix is equal to $\nabla^2 U$. But for stability, we require each of the diagonal elements to be positive.

Stability Conditions

- Thus, we have our three stability conditions:

$$-\nabla U = 0$$

$$\frac{\partial^2 U}{\partial x^2} > 0, \ \frac{\partial^2 U}{\partial y^2} > 0, \ \frac{\partial^2 U}{\partial z^2} > 0$$

- If this is true, then:

$$\nabla^2 U > 0$$

- Note: this does not work the other way round, however if $\nabla^2 U \leq 0$, then the diagonal elements cannot be simultaneously positive.

Electrostatics

- We can link all of these conditions using the following:

$$\nabla \cdot \overrightarrow{E} = \nabla \cdot (-\nabla U) = -\nabla^2 U$$

- We have a solution for that: $\nabla \cdot \overrightarrow{E} = \dfrac{\rho}{\varepsilon_0}$
- Free space has no net charge density, thus $\rho = 0$
- Therefore, $\nabla^2 U = 0$, meaning it is a saddle point
- Thus, <u>it cannot stably levitate!</u>

Electrostatics

- Going one step further, $\nabla^2 U = 0$ satisfies Laplace's function, which is a harmonic function.
- Harmonic functions have a mean value property, meaning the value of the function at any point is equal to the average of all of the points in a sphere around it.
- If that point is an extremum, then you cannot get the value of the function at that point by taking the average of points whose values are all equal to zero.
- Thus, we can prove that $\nabla^2 U = 0$ over *ALL SPACE*, meaning it cannot occur anywhere!

Conclusion: Electrostatic Levitation cannot exist!

Magnetostatics

- For magnets, we use the same stability conditions.
- The potential energy of a magnetic dipole in a magnetic field is:

$$U = -\overrightarrow{\mu} \cdot \overrightarrow{B}$$

- For a permanent magnet, $\overrightarrow{\mu}$ is uniform, so, applying the Laplacian:

$$\nabla^2 U = -\mu \nabla^2 \overrightarrow{B}$$

- The Laplacian has an identity that contains things we already know:

$$\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)$$

Magnetostatics

$$\nabla^2 \overrightarrow{B} = \nabla(\nabla \cdot \overrightarrow{B}) - \nabla \times (\nabla \times \overrightarrow{B})$$

Gauss' Law for magnetism

$$\nabla \cdot \overrightarrow{B} = 0$$

Maxwell-Faraday Equation

$$\nabla \times \overrightarrow{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} \right)$$

Magnetostatics

- We know that in this case, where there is no electric field, $\frac{\partial \overrightarrow{E}}{\partial t} = 0$
- In free space $\vec{J} = 0$, so...

$$\nabla \times \overrightarrow{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} \right) = 0$$

- Thus, $\nabla^2 U = 0$ for permanent magnets in magnetic fields too!

Congrats!

You have just proved Earnshaw's Theorem

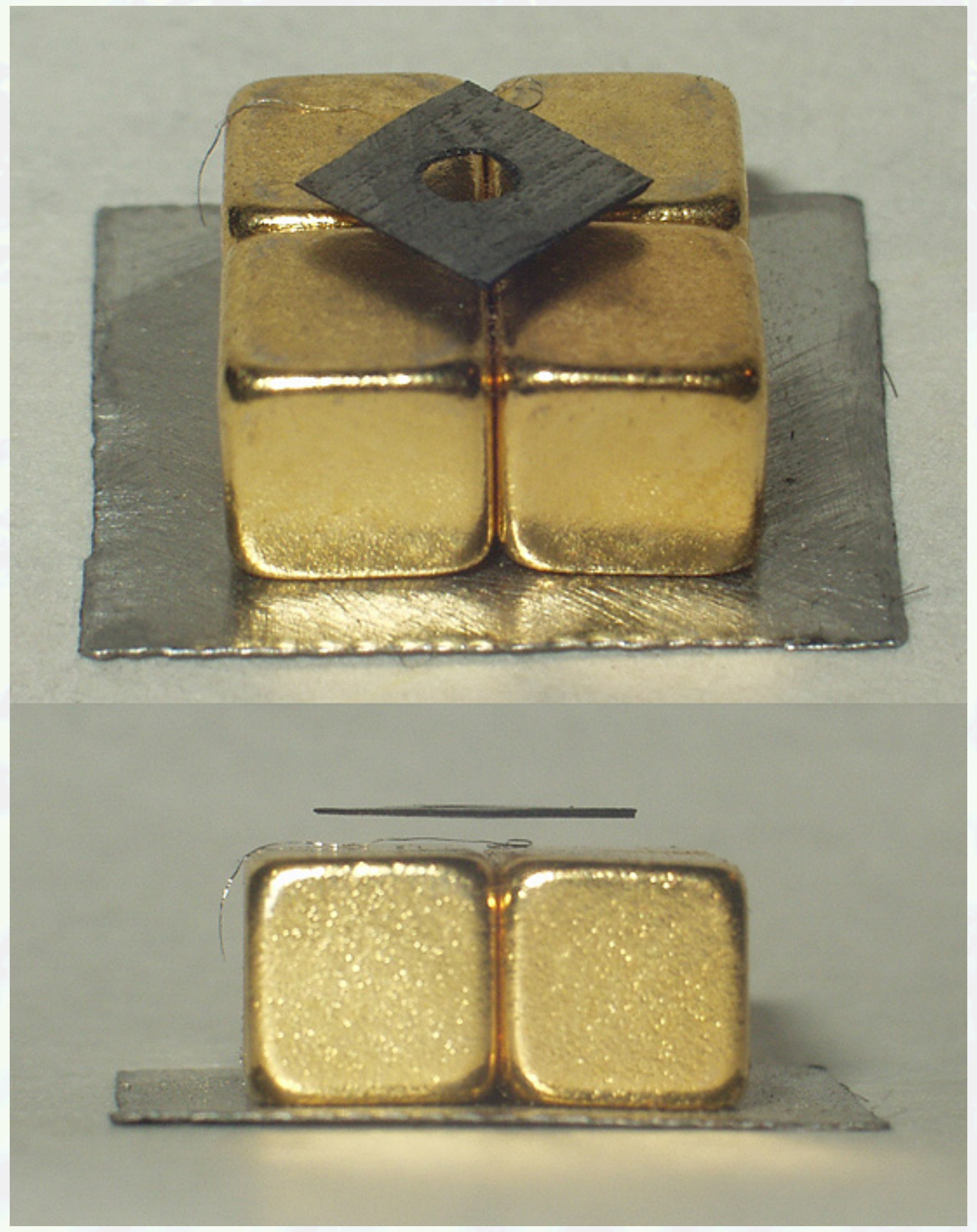
Earnshaw's Theorem
(With a tiny bit of history)

- Samuel Earnshaw formulated his theorem in 1842, stating:

"a collection of point charges cannot be maintained in a stable stationary equilibrium configuration solely by the electrostatic interaction of the charges."

But it does exist!

- *HOWEVER,* we know that electromagnetic levitation does exist in some regard, because we see it in real life, most notably in things like maglev trains.



Exceptions to Earnshaw’s Theorem

- Much of the work finding exceptions to Earnshaw’s Theorem was done by Werner Braunbeck (who I couldn’t find any photos of)
- In 1939, he proved that diamagnets could stably levitate in an applied magnetic field
- For that, we need to know a bit more about magnetic materials…

Magnetic Materials

- There are 2 categories of materials that have no magnetic field in free space, but produce a magnetic field when one is applied to it.
- This means there is a relationship between the magnetisation vector $\overrightarrow{M}$ and the magnetic field strength $\overrightarrow{H}$.
  - Paramagnetic materials produce a magnetic field parallel to the field applied to it.
  - Diamagnetic materials produce a magnetic field normal to the field applied to it.

Magnetic Materials

- We then denote a new variable χ_B as the magnetic susceptibility, which is equal to $\dfrac{\overrightarrow{M}}{\overrightarrow{H}}$
- Paramagnetic materials have positive magnetic susceptibility and diamagnetic materials have negative magnetic susceptibility.
- Now, to use our stability conditions, we need to find a function for the potential energy of a magnetic system

To the Whiteboard!

Again...

Deriving the Potential Energy

- We start by multiplying the Maxwell-Faraday Equation and Ampère's Circuital Law by $\overrightarrow{B}$ and $\overrightarrow{E}$ respectively.

$$\overrightarrow{B} \cdot \nabla \times \overrightarrow{E} = -\overrightarrow{B}\frac{\partial \overrightarrow{B}}{\partial t}$$

$$\overrightarrow{E} \cdot \nabla \times \overrightarrow{B} = \frac{1}{c^2}\overrightarrow{E}\frac{\partial \overrightarrow{E}}{\partial t}$$

Deriving the Potential Energy

- We may now use the triple product identity (sometimes also called the mixed product:

$$a \cdot (b \times c) = -b \cdot (a \times c) = b \cdot (c \times a)$$

- Rearranging for something we can work with:

$$-\nabla \cdot (\vec{E} \times \vec{B}) = \frac{1}{c^2}\frac{1}{2}\frac{\partial |\vec{E}|^2}{\partial t}$$

$$\nabla \cdot (\vec{E} \times \vec{B}) = -\frac{1}{2}\frac{\partial |\vec{B}|^2}{\partial t}$$

- Note: this simplification of the partial derivatives comes from the product rule

Deriving the Potential Energy

- Each of these are the contribution of the $\vec{E}$ and $\vec{B}$ fields, so we can sum them:

$$\nabla \cdot (\vec{E} \times \vec{B}) = \frac{1}{c^2}\frac{1}{2}\frac{\partial |\vec{E}|^2}{\partial t} + \frac{1}{2}\frac{\partial |\vec{B}|^2}{\partial t}$$

- Now, as $\frac{1}{c^2} = \mu_0 \varepsilon_0$, we can substitute this in:

$$\nabla \cdot \left(\frac{\vec{E} \times \vec{B}}{\mu_0} \right) = \frac{\varepsilon_0}{2}\frac{\partial |\vec{E}|^2}{\partial t} + \frac{1}{2\mu_0}\frac{\partial |\vec{B}|^2}{\partial t}$$

- Which is in the form of the continuity equation $\frac{\partial \rho_u}{\partial t} + \nabla \cdot \vec{J}_u = 0$, where ρ_u is the energy density and $\vec{J}_u$ is the energy current density

Deriving the Potential Energy

- Thus, we can find ρ_u to be:

$$\rho_u = \frac{\varepsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2$$

- The potential energy is then:

$$U = \int_V \rho_u \, dV = \int_V \frac{\varepsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \, dV$$

$$= \frac{\varepsilon_0}{2} |\vec{E}|^2 V + \frac{1}{2\mu_0} |\vec{B}|^2 V$$

Deriving the Potential Energy

$$U = \frac{\varepsilon_0}{2} | \vec{E} |^2 V + \frac{1}{2\mu_0} | \vec{B} |^2 V$$

- From this generalised form, as we are dealing with a purely magnetic system, we can disregard the $| \vec{E} |^2$ term
- As we are looking at a magnetic material, we are interested only in the *magnetisation energy*, as for materials without a permanent magnetisation, that is the only potential energy stored in the system (besides gravity)

Deriving the Potential Energy

- We can do this by simply multiplying by the magnetic susceptibility (dimensionless):

$$U_{mag} = \frac{\chi_B}{2\mu_0} |\vec{B}|^2 V$$

- Now we can construct an formula for the total potential energy, in this case, gravity and magnetisation:

$$U_{tot} = mgz - \frac{\chi_B}{2\mu_0} |\vec{B}|^2 V$$

Stability Conditions

- We can use our stability conditions from earlier:

$$-\nabla U = 0$$

$$\frac{\partial^2 U}{\partial x^2} > 0,\ \frac{\partial^2 U}{\partial y^2} > 0,\ \frac{\partial^2 U}{\partial z^2} > 0$$

$$\nabla^2 U > 0$$

Stability Conditions

- This gives us:

$$\nabla U = \frac{mgz}{V} - \left(\frac{\chi_B}{2\mu_0}\right)\nabla |\vec{B}|^2 = 0$$

$$\nabla^2 U = -\left(\frac{\chi_B V}{2\mu_0}\right)\nabla^2 |\vec{B}|^2 > 0$$

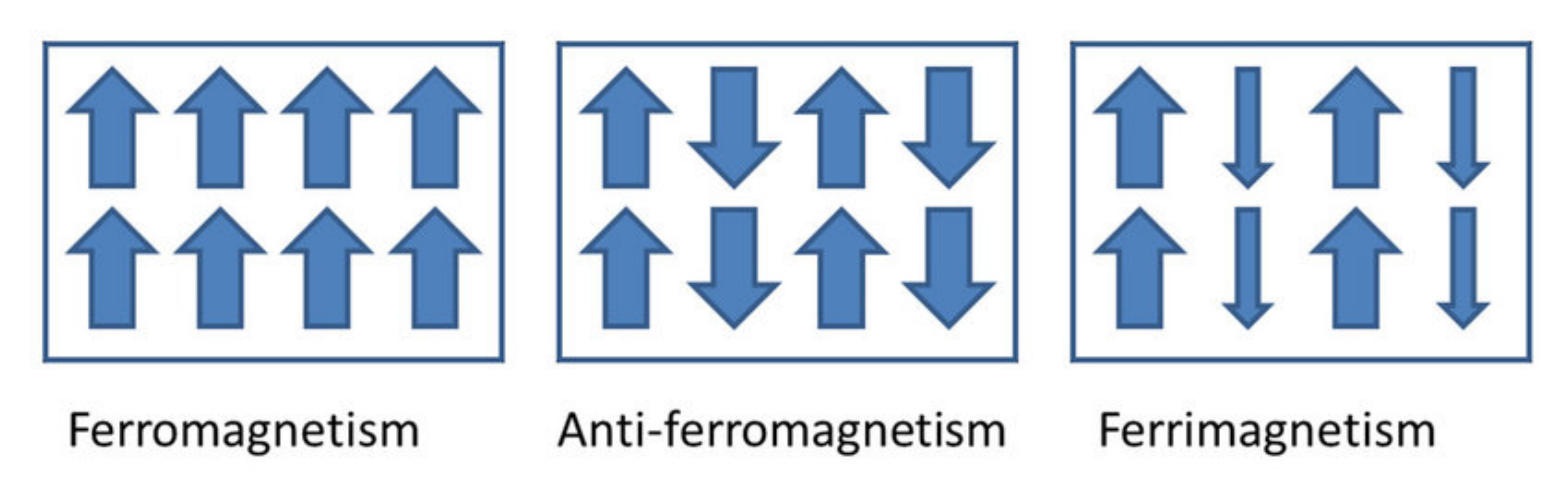
Stability Conditions

- We can quite easily show that the gradient of any squared function is 0 at the y-axis, and that $\nabla^2 |\overrightarrow{B}|^2 > 0$, meaning it is quite possible for ∇U to equal zero.
- For the Laplacian, we know that V and μ_0 are positive constants, thus, as long as χ_B is negative, U satisfies the stability conditions.
- Diamagnets have a negative magnetic susceptibility, so they can stably levitate!
- This was proved in Braunbeck's exception to Earnshaw's Theorem (1939)

This is diamagnetic levitation!

Permanent magnets are tricky...

- Diamagnets and paramagnets have no permanent magnetisation - i.e. they don’t produce a magnetic field until one is applied to them
- For this, we need classify each different type of permanent magnet

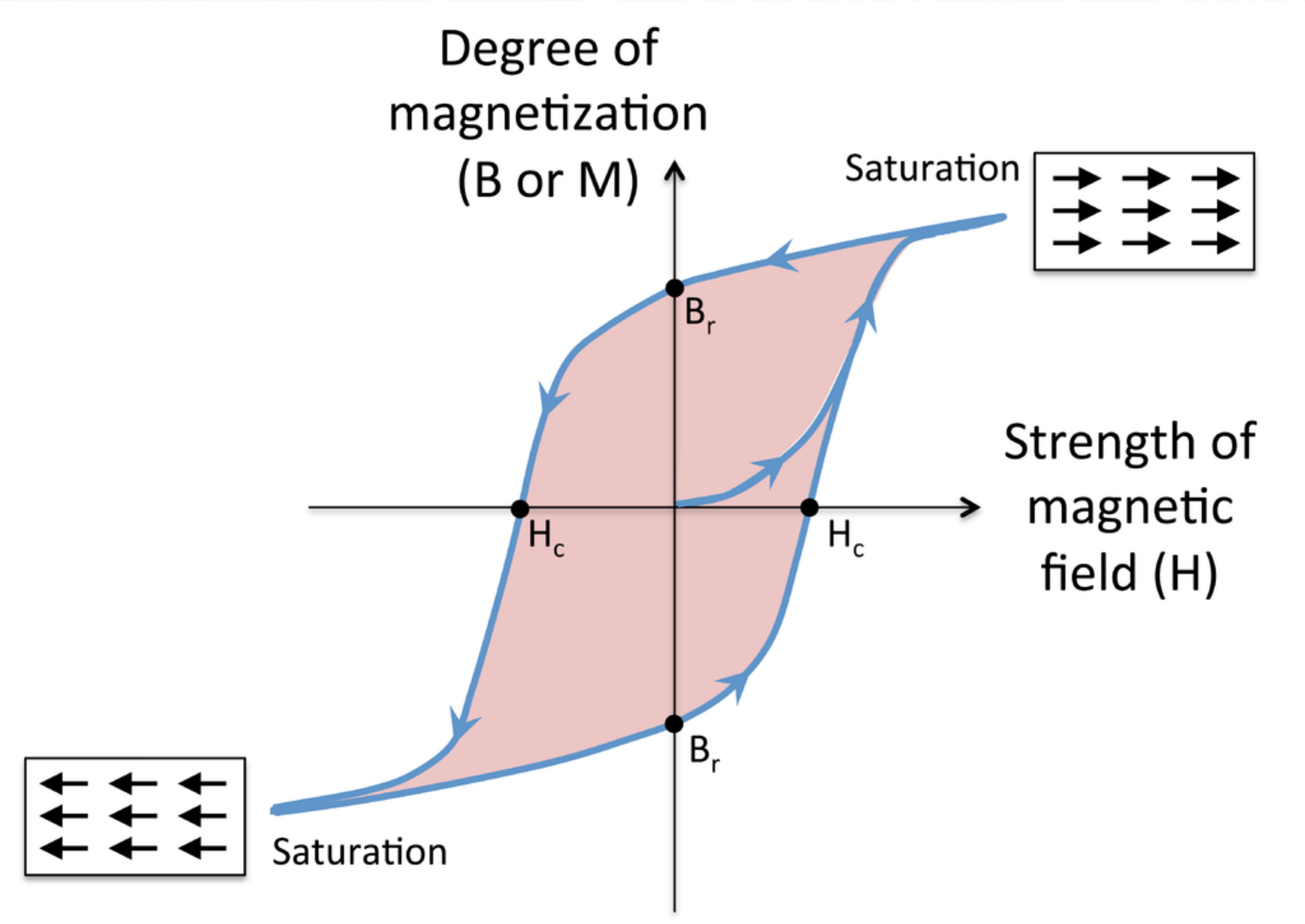


Magnetic Hysteresis

- All of the different permanent magnets all share this property
- As a result of differential magnetic susceptibility

$$\chi_{ij} = \frac{\partial \overrightarrow{M}_i}{\partial \overrightarrow{H}_j}$$

- Forms a loop after initialisation



Magnetic Hysteresis

- As is shown on the plot, the line always has a positive gradient, so we *assume* that the magnetic susceptibility is always positive, in which case, these materials would behave like paramagnets in an applied magnetic field

- A note on the assumption: data on the $\vec{M} - \vec{H}$ curves are empirical, and so we don't have an actual function for it. As such, it is not possible to rule out that there is some permanently magnetised material which aligns its domains normal to the applied field. In this case, we will define these materials (ferromagnetic, ferrimagnetic and antiferromagnetic materials) to have magnetic domains which always align with the applied magnetic field.

Curie/Néel Temperature

- One last thing to consider is the effect of temperature on these materials
- The Curie Temperature is the temperature at which ferromagnetic and ferrimagnetic materials lose permanent magnetisation
- The Néel Temperature is the temperature at which antiferromagnetic materials lose permanent net zero magnetisation.
- The magnetic susceptibility in each of these cases is described by the Curie-Weiss Law:

$$\chi_{ferro} = \frac{C}{T - T_C}, \quad \chi_{antiferro} = \frac{C}{T + T_N}$$

Curie/Néel Temperature

$$C = \frac{\mu_0 \mu_B^2}{3k_B} N_A g^2 J(J+1)$$

- Above the critical temperature, all of these materials behave as though they are paramagnetic, which makes things even simpler!

A summary thus far

Can Levitate

- Diamagnets

Can’t Levitate

- Permanent magnets
- Point charges
- Paramagnets
- Ferromagnets
- Ferrimagnets
- Antiferromagnets

Anything else?

- Feedback systems
- Spin-stabilised magnetic levitation
- Levitation as a result of Lenz' Law
- Pseudo-Levitation

Feedback Systems

- Dynamically adjusts the magnetic field strength depending on position
- Rail similar to a motor which has been “unwrapped” - the linear induction motor
- Dynamic adjustment $\neq$ stable equilibrium

Spin Stabilised Magnetic Levitation

- Small balanced magnet spun at a speed within a physically acceptable range
- Precesses around magnetic field direction
- Most significant factor in this case:

$$\nabla U = \frac{mg}{\mu} - \frac{\partial B_z}{\partial z}$$

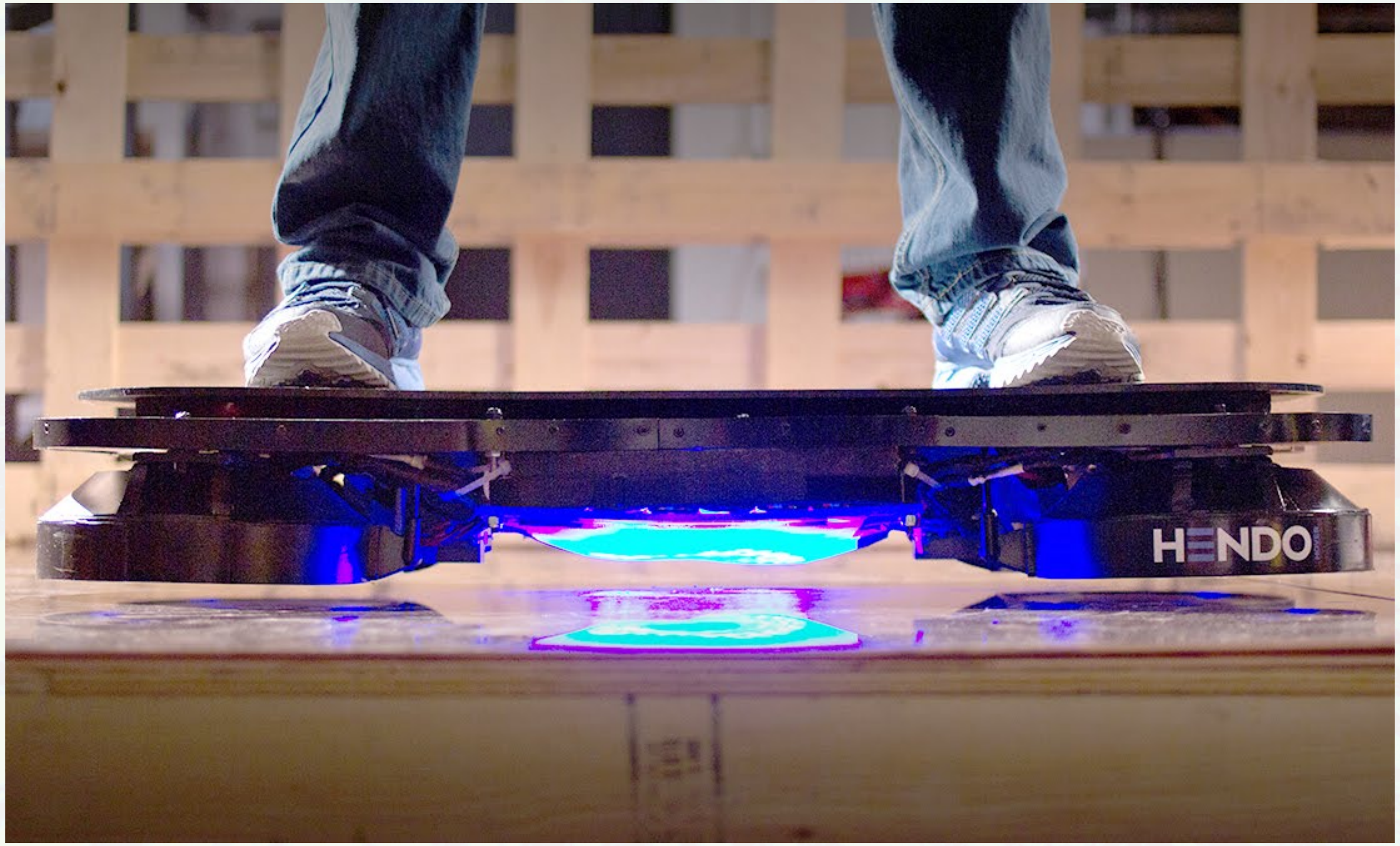
Lenz' Law

- Lenz' Law states that:

"The current induced in a circuit due to a change in a magnetic field is directed to oppose the change in flux and to exert a mechanical force which opposes the motion"

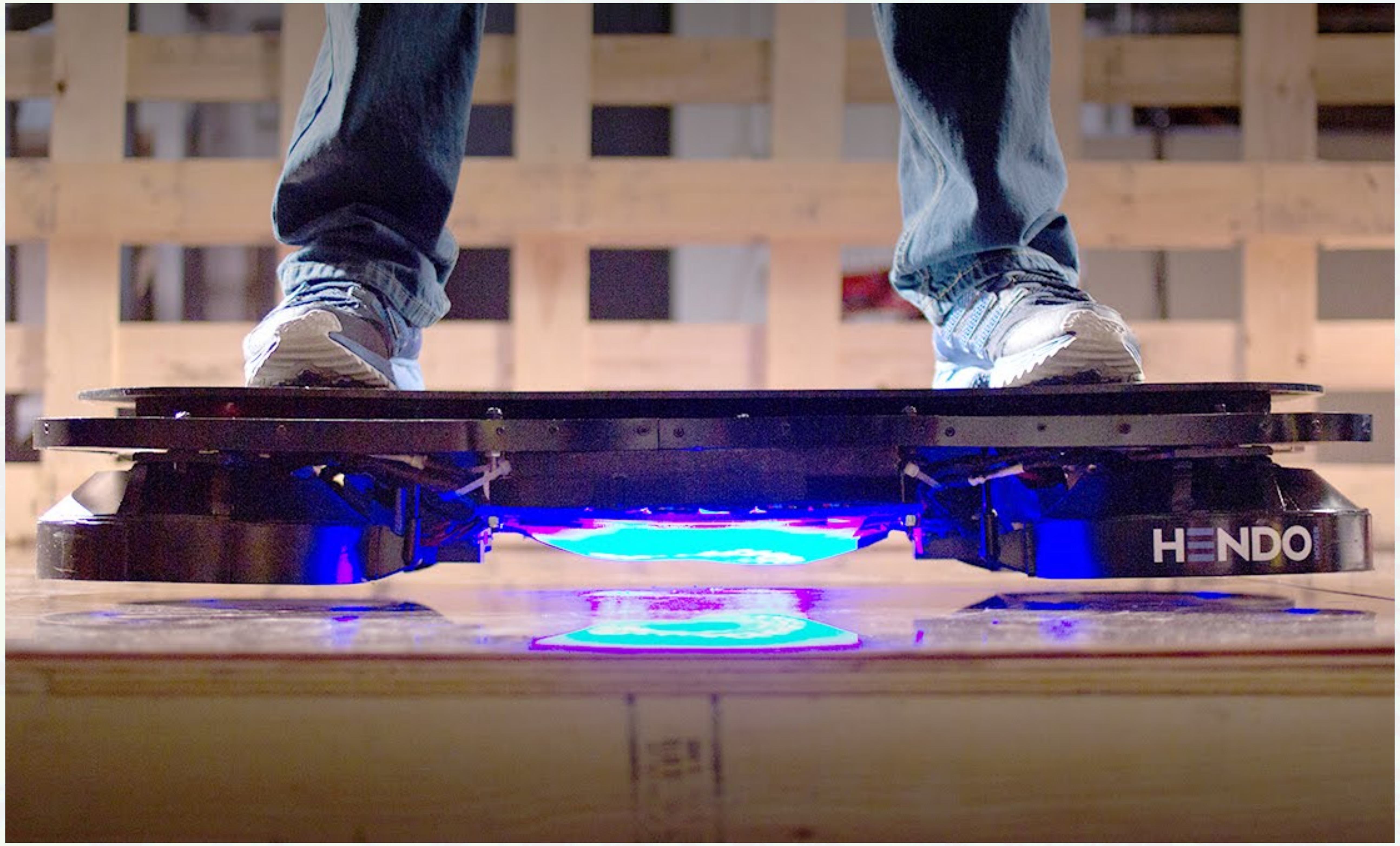
- Or, as an equation from Faraday's Law of Induction:

$$\varepsilon = -\frac{d\Phi_B}{dt}$$



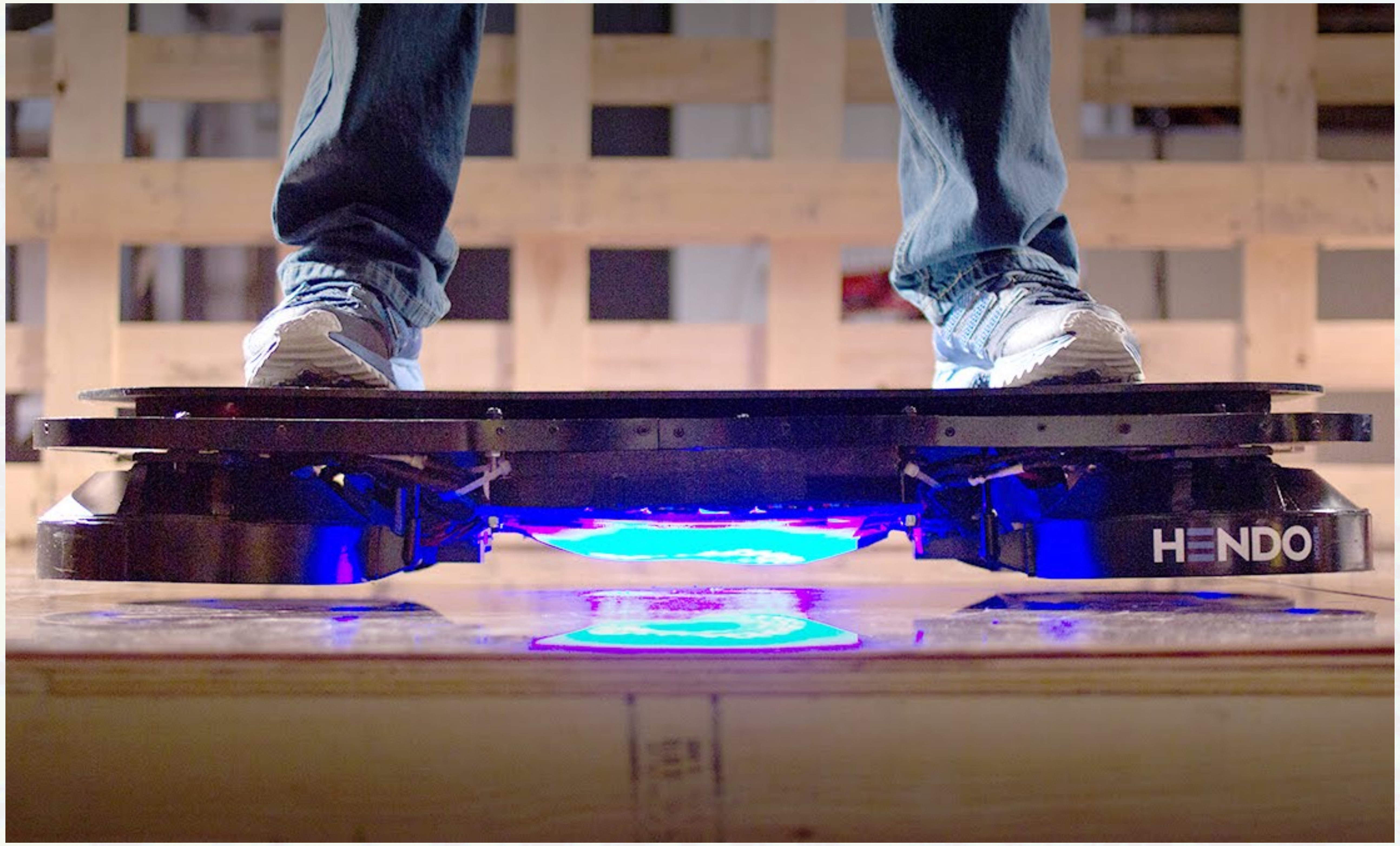
Lenz' Law

- There are two possible systems for this:
  - Applying an alternating current to an electromagnet close to a conductor
  - Eddy currents formed in the conductor, which generates a magnetic field opposing the applied field



Lenz' Law

- Second: moving a permanent magnet relative to a conductor
  - Does the same thing, but purely mechanically
  - The more commonly used of the two systems



One you probably didn't think of

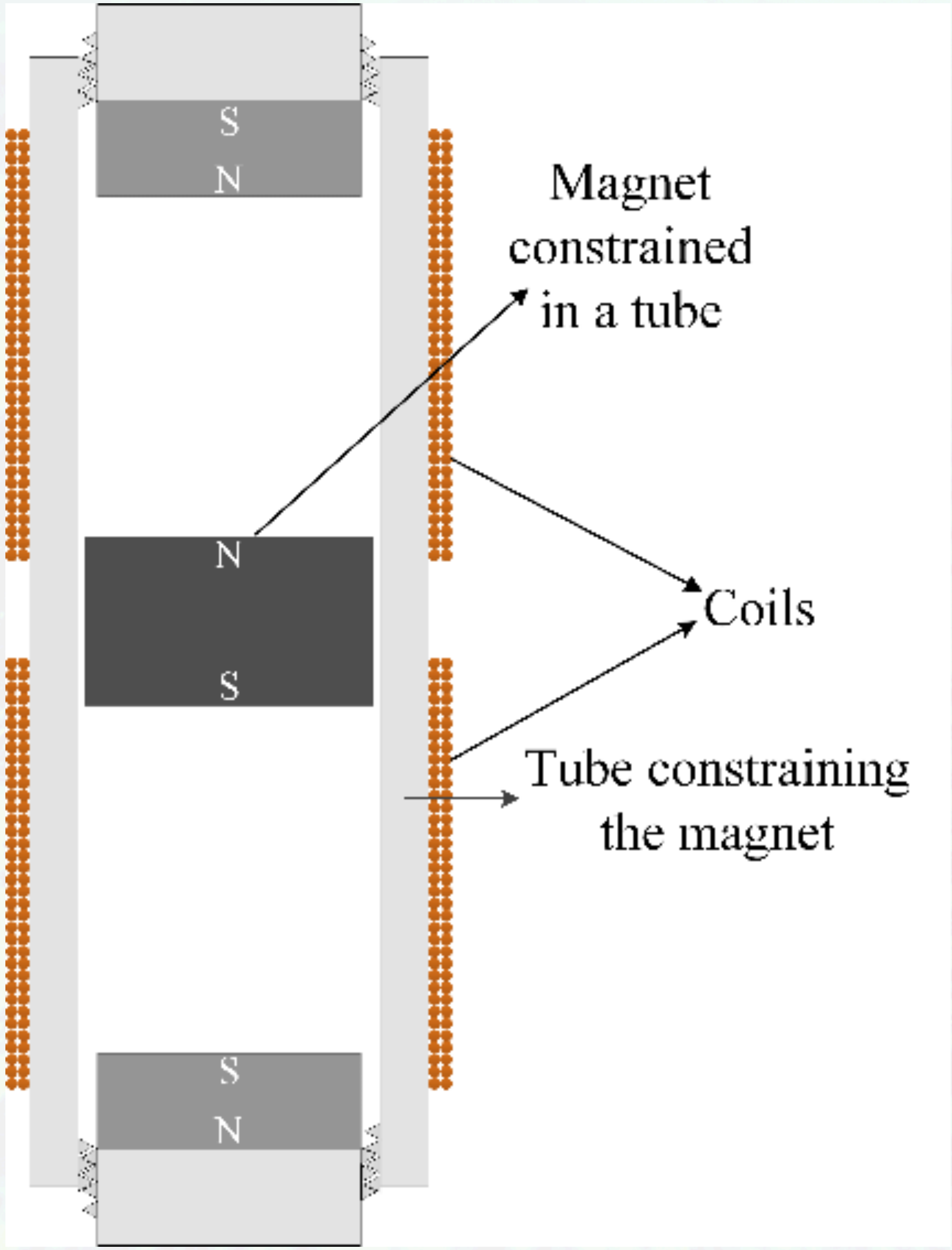
- We proved earlier that the Laplacian of the potential energy of a magnetic system is:

$$\nabla^2 U = -\mu \left(\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2} \right) = 0$$

- Thus, we know that if any of the elements are non-zero, then at least one of them must be negative, meaning the point is unstable in that direction
- Thus, what if we stop the object moving in that direction…?

Pseudo-Levitation

- Now, we constrain the object to only the useful degrees of freedom
- This isn't *really* stable levitation, as we achieve this by constraining the object, however we gain the 'friction-free' benefits of levitation
- These systems have a wide variety of industrial applications



Bonus: a little bit of Quantum

- Atoms *mostly* interact via the Coulomb interaction, so does Earnshaw's theorem hold for this too?
- Let's do a quick, ***simplified*** proof using a little bit of quantum mechanics

- A note on the assumption: we are assuming here that atoms *only* interact via the Coulomb interaction, which is, of course, factually not the case. This proof is to illustrate the behaviours of these systems, and if we can prove that a simple model will not work, then it is easy to assume that more complex systems with additional forces will also not work.

Quantum Mechanics

- Earnshaw's Theorem applies to point charges, but we know that all matter exhibits wave-like properties, which we must consider on this scale:

$$\lambda = \frac{h}{p}$$

- We describe the position of these particles using a wavefunction Ψ.
- These particles are subject to Heisenberg's Uncertainty Principle:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Quantum Mechanics

- So, in order to treat a particle with a wavefunction Ψ as a point mass, we must decrease σ_x to zero - we need to know *exactly* where the particle will be.
- To conserve proportionality, σ_p must then tend towards infinity.
- p and the kinetic energy are linked by the following relation:

$$E_k = \frac{p^2}{2m}$$

Quantum Mechanics

- If σ_p tends towards infinity, so to does σ_{E_k}
- This implies then that the potential energy must be infinite
  - 'Reductio ad absurdum'
- Thus, Earnshaw's theorem must not apply to quantum-scale systems!

So to conclude...

<u>Can Levitate</u>

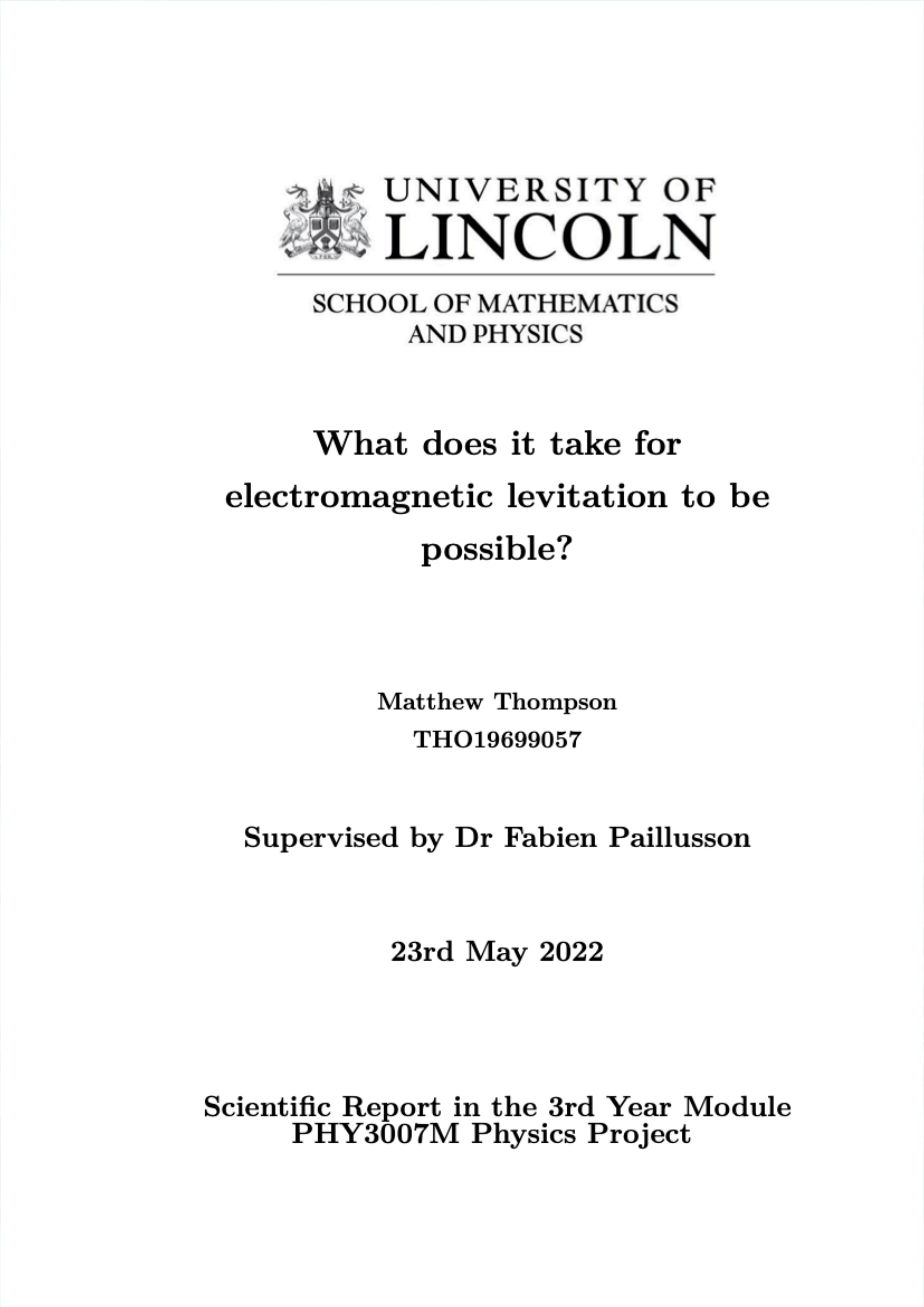
- Diamagnets
- Spin-stabilised systems
- Levitation as a result of Lenz' Law
- Feedback Systems

<u>Can't Levitate</u>

- Permanent magnets
- Point charges
- Paramagnets
- Ferromagnets
- Ferrimagnets
- Antiferromagnets
- Quantum-scale systems*

Fun Fact:

This was my third year project!



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What does it take for electromagnetic levitation to be possible?

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23rd May 2022

Scientific Report in the 3rd Year Module PHY3007M Physics Project

Any questions?